BRAIN POWER
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2017-2018 Math/Logic Contest

For grades 9 – 10 (and up)

Solutions

1. How many zeroes does the number

$$1 \cdot 2 \cdot 3 \cdot \ldots \cdot 100$$

end with?

Answer: 24

Solution. There are 24 fives in the prime factorization of the number.

2. Find the smallest integer that begins with 2016 and is divisible by 2017.

Answer: 20161932

Solution: Since $2017 \cdot 10^k$ is divisible by 2017, and 2017 is a prime, there are no multiples of 2017 in the range between:

$$(2017 \cdot 10^k - 2017) \; and \; 2017 \cdot 10^k$$

For $k < 4$, this range will include all $(4 + k)$ −digit numbers that begin with 2016.

If $k = 4$, we look for the larger m such that $(2017 \cdot 10^k - m \cdot 2017)$ begins with 2016.

We can easily see that $m = 4$.

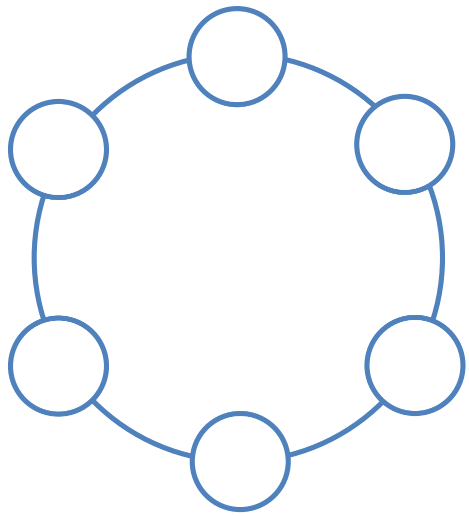
3. Show that the product of four consecutive integers is one less than a perfect square. (For example, $3 \cdot 4 \cdot 5 \cdot 6 = 360$ is one less than 19^2).

Solution. Let $n, n+1, n+2, n+3$ be four consecutive integers. Then

$$n(n+1)(n+2)(n+3) = [n(n+3)] \cdot [(n+1)(n+2)]$$
$$= (n^2+3n)(n^2+3n+2) = (n^2+3n+1)^2 - 1,$$

which is 1 less than $(n^2+3n+1)^2$.

4. In the diagram below, six smaller circles are colored either black or white. Show that no matter which circles are colored black and which are colored white, one of the six circles will have two neighbors of the same color.



Solution. Let's number the circles in a clockwise order: C_1, C_2, C_3, C_4, C_5, C_6. Consider the following group of circles:

(C_1, C_3, C_5)

By Pigeonhole principle, two of these circles will have the same color. But this means that one of the circles (C_2, C_4, C_6) will have both neighbors of the same color:

C_2, neighbors are C_1 and C_3
C_4, neighbors are C_3 and C_5
C_6, neighbors are C_1 and C_5.

5. 11 chess players attend a special tournament where any two players meet in a game at most once (may not meet at all). In the tournament, the first contestant played one game, the second contestant played two games, and so on, the tenth contestant played ten games. How many games did the 11^{th} contestant play?

Answer: 5 games
Solution.
10^{th} player: Since 10^{th} player played 10 games, they played with everyone else, in particular, with the 1^{st} player.
1^{st} player: The 1^{st} player played only one game, and hence, no one else met in a game with them.
9^{th} player: Now, 9^{th} player played 9 games, and since they did not play with the 1^{st} player, they had to play with 2^{nd}, 3^{rd}, …, 11^{th} player.
2^{nd} player: They played with 10^{th} and 9^{th} players, and hence did not play with anyone else.

…
6^{th} player met in a game with 5^{th}, 7^{th}, 8^{th}, 9^{th}, 10^{th}, and 11^{th} players.
5^{th} player met in a game with 10^{th}, 9^{th}, 8^{th}, 7^{th}, 6^{th} players.

Therefore, in total 11^{th} players participated in 5 games (vs 10^{th}, 9^{th}, 8^{th}, 7^{th} and 6^{th} players).

6. Given an equilateral $\triangle ABC$, prove that the sum of distances from any interior point to the sides of an equilateral triangle is always constant.

Solution. Count twice the area of $\triangle ABC$.

7. Below, different letters substitute different digits. It is known that the number

TWOTEN + TWO + TWO

is divisible by 107. Show that the number

TENTWO + TEN + TEN

cannot be divisible by 107.

Note 1: if we read the first number literally we do get that (210 + 2 + 2) is divisible by 107 ☺

Note 2: an example of letter assignment that makes the first statement true: $124193 + 124 + 124 = 107 \cdot 1163$.

Solution. Consider the difference between the two numbers:

$$(TWOTEN + TWO + TWO) - (TENTWO + TEN + TEN) = 1001 \cdot (TWO - TEN)$$

107 does not divide 1001 neither it divides (TWO – TEN) as the latter is only a 2-digit number.