2017-2018 Math/Logic Contest

For grades 7 – 8

Solutions

1. Find all four-digit numbers that begin with 82 and are divisible by 45.

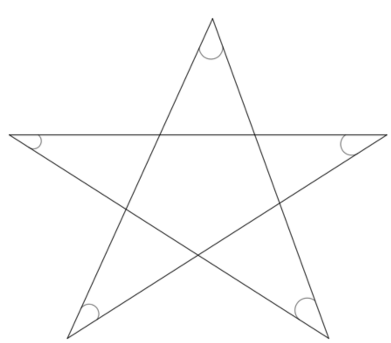
   Answer: 8235, 8280.
   Solution.
   The numbers have to end by 5 or 0, and sum of the digits must be divisible by 9.

2. For the potluck party, students in the class brought cookies and candies. Each girl baked 18 cookies while each boy bought 11 candies. At the party, each girl ate 7 different treats and each boy had 21, and none of the goodies was left over. Were there more girls or boys in the class?

   Answer: boys
   Solution. Let x be the number of girls in the class, and y be the number of boys in the class. Then
   $$18x + 11y = 7x + 21y,$$
   or,
   $$11x = 10y.$$

   Clearly, $x < y$.

3. Find the sum of the interior angles of the star pentagram below:



   Answer: 180 $degrees$.
   Solution. Apply Exterior angle theorem twice.

4. Let n be a positive integer. If $\gcd(n, 2000) = 4$ and $\gcd(n+1, 2000) = 5$, what is $\gcd(n+2, 2000)$?
(gcd(a, b) stands for the greatest common divisor of two integers a and b)

Answer: 2
Solution. Since $2000 = 2^4 \cdot 5^3$, $\gcd(n+2, 2000)$ will have a form

$$2^m \cdot 5^k, \quad m \leq 4 \; and \; k \leq 3.$$

Since $\gcd(n+1, 2000) = 5$, $n+2$ cannot be divisible by 5, and so $k = 0$. Since $\gcd(n, 2000) = 4$, n must be a multiple of 4, and so $n+2$ can only be divisible by 2 and not 4, and so $m = 1$.

5. Below, different letters substitute different digits:

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  F O R T Y
+
      T E N
+
      T E N
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  S I X T Y
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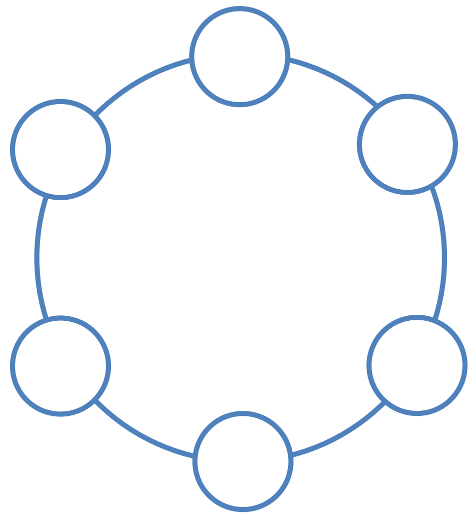
Find which digits are encoded in this addition puzzle.

Answer:
Solution: It is not hard to see that N = 0, E = 5, I = 1, O = 9, S = F + 1. Therefore, 9 > X, T, R > 1, and R + 2T +1 > 21. From that, T > 6 and R > 5. If T = 7, then R = 8 and X = 3. But then we would not find a pair of consecutive digits for F and S (only 2, 4, 6 left). Hence, T = 8, R = 7, X = 4, and so, F = 2, S = 3 and Y = 6.

6. In the diagram below, six smaller circles are colored either black or white. Show that no matter which circles are colored black and which are colored white, one of the six circles will have two neighbors of the same color.



Solution. Let's number the circles in a clockwise order: C_1, C_2, C_3, C_4, C_5, C_6. Consider the following group of circles:

(C_1, C_3, C_5)

By Pigeonhole principle, two of these circles will have the same color. But this means that one of the circles (C_2, C_4, C_6) will have both neighbors of the same color:

C_2, neighbors are C_1 and C_3
C_4, neighbors are C_3 and C_5
C_6, neighbors are C_1 and C_5.